Test and Experimentation Designs

I. ANALYSIS OF VARIANCE

Generally, OT&E and Experimentation involve the comparison of more than two population means based on sample statistics. The technique usually employed to solve such problems is called ANALYSIS OF VARIANCE. Fundamentally, analysis of variance (hereafter, abbreviated ANOVA) is just what the name implies - partitioning the variance of the dependent variable from an experiment into parts to test whether or not certain factors (independent variables) that were introduced into the design actually affect its value. For example, is the miss-distance of a missile system affected by the particular aircraft which releases it? Does the type of radar aboard an aircraft affect the time of target acquisition? In each case, there is interest in testing whether the factor(s) under study significantly affect the measured response variable when compared to the random variation in the process.

When the proper conditions and assumptions are present, the ANOVA technique is a powerful technique to use for statistical problems that involve the comparison of more than two means. Basically, the efficiency of ANOVA is derived by utilizing all the observations across all combinations of test factors to estimate the EXPERIMENTAL ERROR or random error inherent in the process. The F-test or variance ratio is used to compare the estimated variability attributable to a test factor to the estimated error and, subsequently, test for a significant effect.

Various ANOVA Models. There are numerous types of ANOVA models, each incorporates particular assumptions that describe the manner in which the test is structured and conducted. An overview of several of the principal types of ANOVA models are discussed below:

A. **Single-Factor**. In discussing the single factor ANOVA model, many of the principles involved apply to more complex designs with only slight, but important, modifications. The SINGLE-FACTOR EXPERIMENT involves the test to see whether there is a significant difference between the levels of one factor. For example, consider the experiment where there is interest in determining whether there is a difference between four aircraft types for their effect on the radial miss-distance of an identically-launched missile at a target. Table 1 shows the data for this example.

TABLE 1. RADIAL MISS-DISTANCE

Aircraft Type

I	II	III	IV
51	59	40	37
50	56	41	34
57	45	40	40
54	50	34	38
55	51	38	36

The order in which the 20 observations were taken was completely random. Therefore, the example problem has a COMPLETELY RANDOMIZED DESIGN. The model for the design is given by Equation (1):

$$\mathbf{Y_{ij} = \mu + \alpha_j + \varepsilon_{ij}} \qquad (1)$$

where Y_{ij} is the value for the dependent variable for the i^{th} observation within the j^{th} aircraft type, α_j represents the effect for the j^{th} aircraft type, and ε_{ij} represents the random error that is the present in the i^{th} observation within the j^{th} aircraft type. The model has the following assumptions:

(1) The effects are additive as shown by Equation (1).

(2) The error term ε_{ij} is a Normally and independently distributed random effect. It has mean value zero and its variance is the same for each level (the four aircraft types in the example) of the test factor.

(3) μ is a fixed (but unknown) parameter for the population means.

(4) The sum of the factor level effects add to zero. This may be expressed by Equation (2):

$$\sum_{j=1}^{J} \alpha_j = 0 \qquad \textbf{(2)}$$

where; J = 4 (the number of aircraft types) in our example.

If the J levels of the factor are chosen at RANDOM, the α_j are assumed Normally and independently distributed with mean zero and with a common variance σ_α^2. When the α_j are set at pre-determined levels, they are called FIXED and Equation (2) also applies.

The hypothesis that is tested in the single factor design is H_o:$\alpha_j = 0$ for all j. If this hypothesis is not rejected, then it is assumed that there is no effect introduced by the type of aircraft and that each observation Y_{ij} is made up of a mean μ and a random error ε_{ij}.

The ANOVA TABLE for the example is shown in Table 2. The sources of variation are the "between aircraft types effects" and the experimental error. The SUM OF SQUARES are computed from a set of equations that are derived from the four assumptions for the model and the observation data Y_{ij}. The "degrees of freedom" correspond to the number of observations in a sample used to estimate a parameter minus the number of parameters that are being estimated for the same sample, (e.g., there are four α_j means to estimate the average α effect.) MEAN SQUARES is computed by dividing the sum of squares by the degrees of freedom in each row.

TABLE 2 - SINGLE FACTOR ANOVA EXAMPLE

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Squares	F	F $\alpha = .05$
Between aircraft types, α_j	1135.0	4-1=3	378.3	29.8	3.24
Experimental error, ε_{ij}	203.2	20-3-1=16	12.7		
Totals	1338.2	20-1=19			

The test statistic is the F value which is the ratio of mean squares for α_j and ε_{ij}. The critical region in our example is the range of F values that are larger than the table F value for $\alpha = 0.05$ and 3 degrees of freedom in the numerator and 16 degrees of freedom in the denominator. Since $F = 29.8 > 3.24$, there is at least one statistically significant difference in aircraft types. A casual look at the data in Table 1 re-confirms this statistical decision. The determination of which combinations of aircraft types are different from each other is discussed in a later section (after ANOVA).

B. **Two Factors**. In an effort to further refine the experimental error (which is the yardstick by which to test for a significant effect from the levels of the test factors), a restriction may be added to the randomization in our single factor example problem model. The "restriction" is to consider the effect that different aircrews may have on the measured variable. Table 3 shows how the test would look if the restriction is made that every aircraft type must be used once by each of the five separate aircrews. The result of adding the restriction for aircrews is that the design is now a two factor design. Equation (3) gives the model for our example with a second factor added:

$$\mathbf{Y_{ij} = \mu + \alpha_j + \beta_i + \varepsilon_{ij}} \qquad (3)$$

Where:

β_i represents the aircrew effect.

TABLE 3 - TWO FACTOR DESIGN EXAMPLE PROBLEM

Aircrew, β_i	Aircraft Type, α_j			
	I	II	III	IV
1	43 (Y_{11})	41 (Y_{12})	38 (Y_{13})	41 (Y_{14})
2	41 (Y_{21})	45 (Y_{22})	43 (Y_{23})	37 (Y_{24})
3	53 (Y_{31})	50 (Y_{32})	45 (Y_{33})	44 (Y_{34})
4	52 (Y_{41})	53 (Y_{42})	45 (Y_{43})	44 (Y_{44})
5	55 (Y_{51})	51 (Y_{52})	48 (Y_{53})	47 (Y_{54})

Another way that the design may be described is to refer to the aircrews as "BLOCKS" and that the randomization is now restricted within blocks, (e.g., each aircrew must use each aircraft type but the test order is selected in a random manner). Thus, another name for the design is "SINGLE FACTOR RANDOMIZED COMPLETE BLOCK DESIGN."

The arrangement for the ANOVA table for our two factor example problem is shown in Table 4. One can see that our original error term in the single factor design has now been broken down into two components. If there is a significant aircrew effect, then the original error term would have been large and there probably would have been difficulty in detecting any significant aircraft type level.

TABLE 4 - TWO FACTOR ANOVA EXAMPLE

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Squares	F	$F_{.05}$
Between aircraft types, α_j	SS_α	4 - 1 = 3 (a - 1)	SS_α/3-MS	MS_α/MS_ε	3.49
Between aircrews, β_1	SS_β	5-1=4 (b - 1)	$SS_\beta/4=MS_\beta$	MS_β/MS_ε	3.26
Experimental error, ε_{ij}	SS_ε	20-7-1=12 (a - 1)(b - 1)	$SS_\varepsilon/12=MS_\varepsilon$		
Totals	SS_T	19 ab - 1			

A comparison between our single factor design and the two factor design illustrates an important principle in the DESIGN OF EXPERIMENTS. In the single factor design, the aircrew effect is not in the design model; hence, the aircrew effect is confounded into the experimental error. (This is acceptable when from previous experience there is strong reason to believe that the effect is negligible). In the two factor design, the experimental error is more precisely estimated, (since the potential aircrew effect is identified) but has a corresponding decrease in the number of degrees of freedom, (i.e., 16 in the single factor example versus 12 in the two factor example).

C. **Latin Square**. A Latin Square design is one where each level of each factor is combined only once with each level of two other factors. Consider our previous two factor design example and add a third factor (γ_k=1,2,3,4) that is the four different production lot groups from which are sent the missiles that are used in the test. Further, to illustrate the Latin Square design, the number of levels of aircrews has been reduced from five to four. Table 5 shows the resultant 4 X 4 Latin Square design arrangement where each level of each factor occurs once and only once with each level of each of the other two factors.

Some of the freedom for randomization has been lost with the Latin Square design but not all of it. For a given problem, one can select at random from tables that contain different Latin Squares design arrangements of the required size, (e.g., 4 x 4, 5 x 5).

A serious consideration that must be given to the Latin Square design is that the interaction effects of the test factors are confounded into the experimental error term. If there are interaction effects present in the test, the error term will be inflated and, thus, it will be difficult to detect other significant factor effects. Hence, a critical decision must be prior to the selection of a Latin Square design for a test as to whether there is a strong likelihood that interaction effects will be present.

TABLE 5 - 4 x 4 LATIN SQUARE DESIGN EXAMPLE

	AIRCRAFT TYPES, α_j			
Aircrews, β_l	I	II	III	IV
1	γ_1	γ_2	γ_3	γ_4
2	γ_4	γ_1	γ_2	γ_3
3	γ_3	γ_4	γ_1	γ_2
4	γ_2	γ_3	γ_4	γ_1

Missile Production Lot Groups, γ_k

Equation (4) gives the model for our 4 X 4 Latin Square design example and Table 6

$$\mathbf{Y_{ijk} = \mu + \alpha_j + \beta_i + \gamma_k + \varepsilon_{ijk}} \qquad (4)$$

shows the arrangements for the ANOVA table. The critical region at an $\alpha = 0.05$ is shown in the right column as defined by the $F_{.05}$ with 3 and 6 degrees of freedom in numerator and denominator, respectively.

There are several variations of the basic Latin Square design. A GRAECO-LATIN SQUARE design is one that has four factors each of which has each of its levels appear once and only once with each level of each other factor. Seldom is such a design useful since there are so few degrees of freedom for estimating the experimental error term, and because adding factors increases the number of interactions that must be assumed negligible.

A Latin Square design that doesn't have the same number of levels for each factor present in the design is called a YOUDEN SQUARE or incomplete Latin Square. Such a design may be appropriate if there is a logical reason for a fewer number of levels for a factor, (e.g., there are only three aircraft types of test while there are four groups of aircrews and missile production lot groups).

TABLE 6 - 4 x 4 LATIN SQUARE ANOVA EXAMPLE

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Squares	Fs	$F_{.05}$
Between aircraft types, α_j	SS_α	4-1 = 3	$SS_\alpha/3=MS_\alpha$	MS_α/MS_ε	4.76
Between aircrews, β_i	SS_β	4-1 = 3	$SS_\beta/3=MS_\beta$	MS_β/MS_ε	4.76
Between missile groups, γ_k	SS_γ	4-1 = 3	$SS_\gamma/3=MS_\gamma$	MS_γ/MS_ε	4.76
Experimental error, ε_{ijk}	SS_ε	16-9-1 = 6	$SS_\varepsilon/6=MS_\varepsilon$		
Totals	SS_T	15			

D. **Factorial Designs**.

A FACTORIAL DESIGN is one that has all levels of a given factor combined with all levels of each other factor in the experiment. The factorial design is the most commonly used design in OT&E; the reasons for this popularity are brought out in the following discussion.

Consider an example of a factorial design which has the three factors that have been used in the previous examples: aircraft types, aircrews, and missile production lot groups. The test factor combinations are shown in Table 7.

TABLE 7 - 3 FACTORS, 2 REPLICATIONS FACTORIAL DESIGN EXAMPLE

Aircrew β_i	Aircraft Types, α_j							
	Missile Production Lots γ_k		Missile Production Lots γ_k		Missile Production Lots γ_k		Missile Production Lots γ_k	
	γ_1	γ_2	γ_1	γ_2	γ_1	γ_2	γ_1	γ_2
1	Y Y	Y Y	Y Y	Y Y	Y Y	Y Y	Y Y	Y Y
2	Y Y	Y Y	Y Y	Y Y	Y Y	Y Y	Y Y	Y Y
3	Y Y	Y Y	Y Y	Y Y	Y Y	Y Y	Y Y	Y Y
4	Y Y	Y Y	Y Y	Y Y	Y Y	Y Y	Y Y	Y Y
5	Y Y	Y Y	Y Y	Y Y	Y Y	Y Y	Y Y	Y Y

There are two observations or replications in each cell in the matrix. By taking two or more REPLICATIONS per cell (or combination set of test factors) the interaction effects between test factors may be tested and, also, separated out from the experimental error term. The example presented in the discussion of the two factor design is a factorial design (the Latin Square design may be regarded as a special case of a factorial design); however, there is only one observation per cell and based upon previous knowledge it was assumed that there was no interaction between aircraft types and aircrews. Hence, with this highly restrictive assumption, the previous two factor example problem is not a representative example of the general application of a factorial design.

The model for the factorial design example problem shown in Table 7 is given by Equation (5)

$$\mathbf{Y}_{ijkl} = \mu + \alpha_j + \beta_i + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{jk} + (\beta\gamma)_{ik} + (\alpha\beta\gamma)_{ijk} + \varepsilon_{l(ijk)} \qquad (5)$$

where the terms in parenthesis represent the second and third order interactions between test factors and l(l=1,2) is the subscript that corresponds to the observation number within a cell. Table 8 shows the ANOVA table for the example.

TABLE 8 - FACTORIAL DESIGN EXAMPLE PROBLEM ANOVA TABLE

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Squares	F	$F_{.05}$
Main Effects					
Between aircraft types, α_j	SS_α	4-1=3	$SS_\alpha/3 = MS_\alpha$	MS_α/MS_ε	2.84
Between aircrews, β_i	SS_β	5-1=4	$SS_\beta/4 = MS_\beta$	MS_β/MS_ε	2.61
Between missile groups, γ_k	SS_γ	2-1=1	$SS_\gamma/1 = MS_\gamma$	MS_γ/MS_ε	4.08
Interactions:					
$\alpha \times \beta$	$SS_{\alpha\beta}$	3x4=12	$SS_{\alpha\beta}/12 = MS_{\alpha\beta}$	$MS_{\alpha\beta}/MS_\varepsilon$	2.00
$\alpha \times \gamma$	$SS_{\alpha\gamma}$	3x1=3	$SS_{\alpha\gamma}/3 = MS_{\alpha\gamma}$	$MS_{\%\gamma}/MS_\varepsilon$	2.84
$\beta \times \gamma$	$SS_{\beta\gamma}$	4x1=4	$SS_{\beta\gamma}/4 = MS_{\beta\gamma}$	$MS_{\beta\gamma}/MS_\varepsilon$	2.61
$\alpha \times \beta \times \gamma$	$SS_{\alpha\beta\gamma}$	4x3x1=12	$SS_{\alpha\beta\gamma}/12 = MS_{\alpha\beta\gamma}$	$MS_{\%\beta\gamma}/MS_\varepsilon$	2.00
Experimental Error $\varepsilon_{l(ijk)}$	SS_ε	80-39-1=40	$SS_\varepsilon/40 = MS_\varepsilon$		
Totals	SS_T	79			

The principal reasons why the factorial design is used so often in OT&E are summarized:

(1) Generally, an OT&E test program has ambitions goals to investigate the effect of several test factors on the response variable. The factorial design permits the most efficient method to test several test factors (compared to a series of one-factor-at-a-time experiments).

(2) Every observation is used to estimate an effect from each test factor. Some level of each test factor is present in each observation.

(3) The experimental error is estimated over a wide range of test conditions and, generally, there is an adequate sample size (degrees of freedom) available for its estimation.

(4) When there are two or more observations per cell, an isolation and estimation of possible interaction effects between test factors may be performed.

E. **Other Designs**.

There are numerous other ANOVA designs that may be applied to OT&E test programs. It is beyond the scope of this document to discuss the other designs in comprehensive detail. A brief mention is made below of some of the principal designs that are employed in OT&E. The test planner should consult with a statistician for the application of the proper statistical design.

A **FRACTIONAL FACTORIAL** design is a factorial design that has an incomplete number of observations for at least on replication for the design matrix. For example, if one or more of the observations were not available for the factorial design shown in Table 7, the design would be referred to as fractional factorial. Often, limited test resources and/or lack of interest for particular sets of combinations of test factors (cells) leads to fractional factorial test designs.

Whether a test factor is fixed or random will influence the form and interpretation of the F-tests that are made to test for significant effects. When all the levels of each factor in a design are fixed or set at pre-determined levels, the experiment has a fixed model. When all levels of each factor in a design are chosen at random, the test has a random model. When the design involves one or more factors that have their levels fixed and one or more factors that have their levels random, the experiment has a mixed model. In the example problem displayed in Table 7, the aircraft type and missile production lot groups are fixed factors. If the aircrews are chosen at random, then the test has a mixed model.

There are test situations where there are test factors that are not factorial or crossed (taken in combination with) over all levels of each of the other test factors, (i.e., there is a test factor that is nested within, or are sub-samples of, levels of another factor). When an experiment involves (1) test factors that are crossed (factorial) with other test factors and (2) test factors that are nested within levels of other test factors, it is referred to a nested-factorial experiment.

Another example is a **NESTED FACTORIAL** design. Each aircraft type has four missile launch racks. However, the same four launch racks are not used on each aircraft type. Thus, the launch rack effect, $\gamma_{k(j)}$, is nested within the aircraft type factor. It is important to recognize when a design has a nested rather than a factorial test factor since the experiment model and ANOVA table breakdown is different compared to a completely factorial design.

There are many experimental situations where it is impractical to completely randomize the order of taking test observations among the levels of a test factor. For example, consider an experiment where there is interest in testing the performance (in terms of "time-to-target-acquisition") of three difference radar types over four different target types and have three replications for each combination set of test factors. To achieve a completely randomized two factor design, a radar type would be used with one of the target types to be identified and then another radar type/target type combination would be selected at random for the next observation, and so on …. Such an experiment would entail flying for 36 separate combinations of radar type/target conditions.

t – Tables

$\nu\backslash\alpha$	.005	.01	.025	.05	.10	.20	.25	.30	.40	.45
1	63.66	31.82	12.71	6.31	3.08	1.376	1.000	.727	.325	.158
2	9.92	6.96	4.30	2.92	1.89	1.061	.816	.617	.289	.142
3	5.84	4.54	3.18	2.35	1.64	.978	.765	.584	.277	.137
4	4.60	3.75	2.78	2.13	1.53	.941	.741	.569	.271	.134
5	4.03	3.36	2.57	2.02	1.48	.920	.727	.558	.267	.132
6	3.71	3.14	2.45	1.94	1.44	.906	.718	.553	.265	.131
7	3.50	3.00	2.36	1.90	1.42	.896	.711	.549	.263	.130
8	3.36	2.90	2.31	1.86	1.40	.889	.706	.546	.262	.130
9	3.25	2.32	2.26	1.83	1.38	.883	.703	.543	.261	.129
10	3.17	2.76	2.23	1.81	1.37	.879	.700	.542	.260	.129
11	3.11	2.72	2.20	1.80	1.36	.876	.697	.540	.260	.129
12	3.06	2.68	2.18	1.78	1.36	.873	.695	.539	.259	.128
13	3.01	2.65	2.16	1.77	1.35	.870	.694	.538	.259	.128
14	2.98	2.62	2.14	1.76	1.34	.868	.692	.537	.258	.128
15	2.95	2.60	2.13	1.75	1.34	.866	.691	.536	.258	.128
16	2.92	2.58	2.12	1.75	1.34	.865	.690	.535	.258	.128
17	2.90	2.57	2.11	1.74	1.33	.863	.689	.534	.257	.128
18	2.88	2.56	2.10	1.73	1.33	.862	.688	.534	.257	.127
19	2.86	2.54	2.09	1.73	1.33	.861	.688	.533	.257	.127
20	2.84	2.53	2.09	1.72	1.32	.860	.687	.533	.257	.127
21	2.83	2.52	2.08	1.72	1.32	.859	.686	.532	.257	.127
22	2.82	2.51	2.07	1.72	1.32	.858	.686	.532	.256	.127
23	2.81	2.50	2.07	1.71	1.32	.858	.685	.532	.256	.127
24	2.80	2.49	2.06	1.71	1.32	.857	.685	.531	.256	.127
25	2.79	2.48	2.06	1.71	1.32	.856	.684	.531	.256	.127
26	2.78	2.48	2.06	1.71	1.32	.856	.684	.531	.256	.127
27	2.77	2.47	2.05	1.70	1.31	.855	.684	.531	.256	.127
28	2.76	2.47	2.05	1.70	1.31	.855	.683	.530	.256	.127
29	2.76	2.46	2.04	1.70	1.31	.854	.683	.530	.256	.127
30+	2.58	2.33	1.96	1.645	1.28	.842	.674	.524	.253	.126
ν/α	.005	.01	.025	.05	.10	.20	.25	.30	.40	.45

VALUES OF F .05*

V_1 = Degrees of freedom from numerator

V_2 = Degrees of freedom for denominator	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	161	200	216	225	230	234	237	239	241	242	244	246	248	249	250	251	252	253	254
2	18.5	19.0	19.2	19.2	19.3	19.3	19.4	19.4	19.4	19.4	19.4	19.4	19.4	19.5	19.5	19.5	19.5	19.5	19.5
3	10.10	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.37
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.38	2.38	2.30	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	3.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.93
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00